Finite width of shear zones

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We present some experimental and numerical results based on a simple model designed to give an estimate for the width of a shear zone. We conclude that the observed finite size of the shear zone can be associated with the propagation of the force lines inside the medium. The model, based on a simple argument on the force distribution and dilatancy, predicts a width of about ten grain diameters.

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I. INTRODUCTION

Granular materials are important for a wide variety of technical processes. They have unusual properties and one of the major problems in soil mechanics is to predict the properties of sheared granular media. The classical description for such a deformation is given by the so-called nonassociate Mohr-Coulomb plasticity theory as introduced by Drucker and Prager in 1952. Rudnicki and Rice [1] showed that, except for the special case in which the friction angle equals the dilatancy angle, a spatial instability shows up in the form of a mathematical bifurcation, which physically means that, essentially due to strain softening, the deformation rate concentrates on planes. The theory predicts mathematical planes, that is, planes with zero width.

All observations, however, show that the shear bands have a finite width of the order of ten grain diameters. This result has been found experimentally by a direct observation of the shear band [2,3], or numerically [4–8] by direct simulation. This finite width naturally changes the macroscopic mechanical response of the packing as compared to the prediction of the theory. For this reason there has been in recent years strong research activity on the one hand to understand and explain the finite width and on the other hand to incorporate into the existing plasticity theory a tool that would produce shear bands of finite width. Three different approaches have been suggested to introduce this length scale into the continuum theory: nonlocal elasticity, gradient elasticity, and Cosserat elasticity. All of these theories are heuristic and a micromechanical justification is still lacking.

Another domain that is directly concerned with this problem is granular friction. The frictional properties [9,10], which are very important in various domains, are directly governed by the shear bands. As an example, the study of the motion of a fault zone in geophysics demands a good understanding of the friction of the gauge. At present these geophysical systems are essentially described phenomenologically using rate and state constitutive equations [11–17].

The main purpose of this paper is to give a simple description of the shear zone in order to obtain some quantitative information about the width of the mobile zone. We focus on the stress propagation inside the medium assuming that the main properties of the band are governed by stress paths. We want to discuss a model to calculate the finite width of the shear zone from microscopic configurations, i.e., on the level of the grains and their contacts. Previous studies based on continuum models do not permit a good understanding of the underlying physics. As already discussed in classical plasticity theory, these bands have zero width [1] and the discrete nature of the material seems to play an important role in the creation of these bands.

In order to obtain some information about the size and the dynamics of the shear zones, we propose an experiment and a discrete model. Our work is based on a simple assumption: the motion of the beads is governed by the compaction of the medium, which depends on the stress distribution inside the medium. The propagation of the force through the beads goes along a complex structure and these force concentrations can be displayed as lines between grains yielding connected networks of force transmission [18]. Some of the force lines can eventually turn back and dilate the medium. This effect appears clearly in experiments where we push a piston down into sand; the force lines that turn back dilate the medium around the piston and move the surface upwards. This effect can also be visualized photoelastically [19]. We assume that this effect produces a decompaction of the medium, which allows the beads to be displaced.

The paper is organized as follows. In Sec. II we present a simple model, which permits us to evaluate the width of the shear band. In Sec. III we detail the experiment and describe the results. In Sec. IV we introduce the model and compare both the experimental and the numerical results. In Sec. V we summarize the main results of the paper.

II. ON THE WIDTH OF THE SHEAR BAND

Let us consider a box filled with dry sand under hydrostatic pressure, which is sheared (since some time) with constant velocity. The largest relative displacements between grains are concentrated in the shear band [20]. This is possible because in the center of the band the density is sufficiently low (below Reynold's dilatancy) so that the grains have enough space to perform displacements. We assume that the grains are sufficiently undeformable themselves. Due to this mobility of grains at the center, we consider the line in the middle of the shear band to be, to a certain extent, similar to a very flat free surface. Indeed, the granular flow down an inclined chute has many similarities to shear bands and also a finite width of the shear zone at the free surface has been measured [10]. Our argument is valid for both situations; the only difference is the flatness of the surface. We are describing a shear band similar to two chute flows going in opposite

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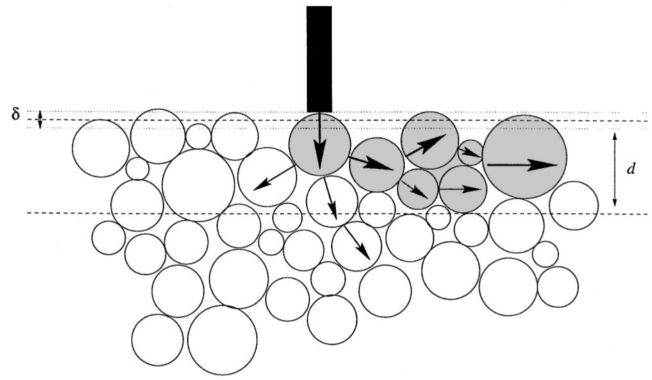

FIG. 1. Experimental protocol. We push one of the disks at the top of the box until a movement takes place and measure the size d of the moving zone. In the experiment the piston can be normal to the surface or orientated at an angle α.

directions being pressed against each other at their free surfaces. Due to the external hydrostatic pressure and the randomness of the motions of the grains not all the grains in the center of the shear zone are mobile and often two grains belonging to opposite sides get stuck against each other, this being the microscopic origin for macroscopic friction. When two grains get stuck they exert big forces on the surrounding medium, which generate some collective displacements of grains that finally unblock the two grains. We argue that due to the nature of force transmission in granular packings these collective displacements are localized in a few layers giving rise to the finite width of the band.

Let us remember that a mechanical property of the granular microstructure is the fact that forces inside the medium are transmitted through essentially pointlike contacts between the grains. These stress concentrations are connected by lines yielding networks of force transmission [18]. Assuming that on average a grain has n contacts, one can quantify the discreteness of the granular structure by the average angle 2β between two contacts, which in two dimensions is $\beta = 180°/n$. When a force is applied on a grain from a certain direction this force is deviated into the directions of the contacts, by an angle of the order of β. Therefore, the propagation of stresses through the packing goes along a kindly structure where at each grain the force lines bifurcate into several branches having a typical angle of β with respect to the original direction. An important consequence for our argument is that due to a sequence of such changes in the direction, these lines can eventually also turn by more than 90°. Once the lines have turned by more than 90°, they have a positive component in the upward direction and can, therefore, mobilize a set of grains. We will, however, in the following, consider as an example only lines that turn by more than 180°, i.e., which all the force shows upward. In the later sections we will, in fact, see that this value is too high. As already mentioned this happens when one pushes a piston down into sand: the sand around the piston displaces upwards. We believe that it is this kind of collective upward motion that in a shear band allows a stuck grain to create itself some space by moving other grains instead towards the center of the band (see Fig. 1).

While there is no doubt that such an effect exists, it remains to be shown that it can account for the mobilization of only about ten layers and thus the finiteness of the shear band. Intuitively one would expect that the turning back of the force lines by 180° is a minor effect since the main stress must go toward the direction of gravity as a continuum theory would predict. But only lines that turn by 180° can mobilize grains (into the less dense center of the band). Such a displacement also releases the stress along this line so that it will not branch further, which, as we will see later, has an important effect on the statistical combinatorics. A simple two-dimensional model should now give us some quantitative answer.

Let us consider a two-dimensional packing of disks of equal size with $n=4$ contacts per particle, which is a typical value for two-dimensional random packings. The deviation angle is therefore $\beta = 45°$. Let us imagine the packing built of layers shifted against each other so that the grains of every second layer are on top of each other. So, we will only give every second layer an integer index in order to introduce the length scale in units of a grain diameter. The grain exerting a force down into the packing be in the zeroth layer. It transmits its force into an intermediate layer in two equal components, deviating them by $\pm 45°$. Within this $\frac{1}{2}$ layer the forces are again split in two by $\pm 45°$ making that half of the force lines go horizontally within this layer. After two further splitting bifurcations, $\frac{1}{8}$ of all force lines already push a grain in the zeroth layer upwards without having touched the layer indexed 1. Next we will calculate the fraction of force lines (or paths) turning upward as a function of the layer depth.

Let us call $\sigma_i = \pm 1$ the variable indicating if a path deviates by $\pm 45°$ at the ith bifurcation. A path going through m grains is characterized by a string $[\sigma_1,\ldots,\sigma_m]$. When the condition $|\sum_{i=0}^{m}\sigma_i| = 4$ is fulfilled, this path has turned upward and is eliminated from the counting since it does not matter how much it might still branch, the force shows upwards and is, therefore, able to mobilize a set of grains. This gives us a truncated Pascal triangle, which can be defined by three quantities: $a(l)$, $b(l)$, and $c(l)$, being the number of paths having made one turn to the right, no turn, and one turn to the left, as a function of the depth l.

Let us define $a(0)=1$, $b(0)=2$, and $c(0)=1$ and iterate as

$$a(l+1) = 2a(l) + b(l),$$

$$b(l+1) = 2b(l) + a(l) + c(l),$$

$$c(l+1) = 2c(l) + b(l).$$

Then one obtains the fraction of paths turning upwards at the lth layer to be

$$o(l) = \frac{\alpha(l) + c(l)}{2^{2l+4}}$$

giving

$$\tfrac{1}{8}, \tfrac{1}{8}, \tfrac{7}{64}, \tfrac{3}{32}, \tfrac{41}{512}, \tfrac{35}{512} \quad \text{for} \quad l = 0,\ldots,5.$$

This means that at the fifth layer, i.e., five grains diameters deep, already 60% $[=\Sigma_{i=0}^{5} o(i)]$ of the possible paths have turned up and could, therefore, have produced a displacement of grains upwards. It is also interesting to remark that the fraction of paths turning upward at level l among those paths that have never reached that level is (expressed in percent)

$$12.5, 14.3, 14.6, 14.6, 14.2, 14.2 \quad \text{for the layers} \quad l=0,...,5.$$

This shows that the process is nearly multiplicative and will, therefore, converge very fast for large l.

It is clear that most of the stress is carried by paths that go downward and that all the paths that turn up together only carry a very small fraction of the total stress [21]. But if the medium is not cohesive, not much force is needed to displace some particles by a little distance. It should be noted that the deeper a path turns upward the larger collective motion of grains it can generate (see shaded region in Fig. 1). Also the larger are the friction forces it has to overcome to mobilize the grains and the smaller is typically the force transmitted.

We can obtain the analytical expression for $o(l)$ in the limit: $1/l \to 0$. The quantities $o(l)$ represent the distribution function over of the total number of steps done before the condition $|\Sigma_{i=0}^{m} \sigma_i| = 4$ is obtained.

We define

$$\xi_m = \sum_{i=0}^{m} \sigma_i , \tag{1}$$

where σ_i is a random variable that takes the values ± 1 with probabilities: $p(-1)=p(1)=0.5$. We can define a stopping time: $\tau = \min(m, |\xi_m| = a)$, where $a=4$ in this case.

The value ξ_m is the discrete version of a one-dimensional Brownian motion. In this limit, ξ_m is a martingale [22] and we can calculate the distribution for τ. The result for the continuum problem is expected to solve our discrete problem in the limit $1/l \to 0$.

Then we can write the expression for $o(l)$,

$$o(l) = 4 \sum_{n=0}^{\infty} (-1)^n \frac{a(2n+1)}{\sqrt{2\pi l^3}} \exp\left\{ \frac{[a(2n+1)]^2}{2l} \right\} \tag{2}$$

with $l=2k$ and $k=2, 3, 4, ...$

Figure (2) shows the result of the model and the result of Eq. (2) with $a=4$.

III. EXPERIMENTS

In order to obtain more detailed information about the dynamics, we made several experiments. We focus on the effect of a pointlike strain submitted locally to a two-dimensional arrangement of disks. We work with a rectangular box made of two vertical plexiglas plates, (see Fig. 3) the space between the two plates is filled with coins. The box is 20 cm high and 55 cm long, the spacing between the two plates is 2 mm. The coins have diameters ranging from 10 to 30 mm. We insert the coins randomly in the box.

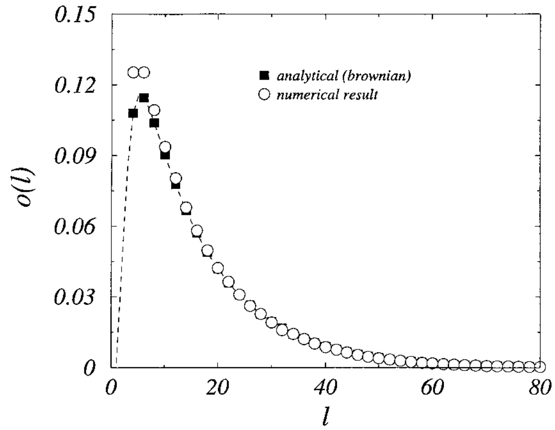

FIG. 2. Comparison between the discrete model and its analytical approximation. The approximation is expected to be good for $l \gg 1$.

Using a metallic piston we apply by hand a tiny displacement on the top of one coin, by increasing the strain until a motion of particles occurs. Then we measure the position of the deepest moving coin and make statistics over different realizations. Each displacement defines an event, which describes the motion of coins. The coins themselves are well identified and we can precisely define the depth. We chose to measure this length as follows: Among the coins that move we identify the deepest and we measure the vertical distance between the bottom of this coin and the top coin on which we applied the pointlike force. This depth corresponds to the distance between the top of the packing and the position where the slide between the mobile and the static zone occurred. Sometimes motion occurs only at the top of the packing due to local instabilities of particles, these events are not considered and are not included in the statistics.

We first prepare the packing of coins using different methods. The first one is obtained by putting the coins randomly inside the medium, without any other action. This is the noncompacted configuration. The second medium consists of the previous random packing but in a compacted form. The com-

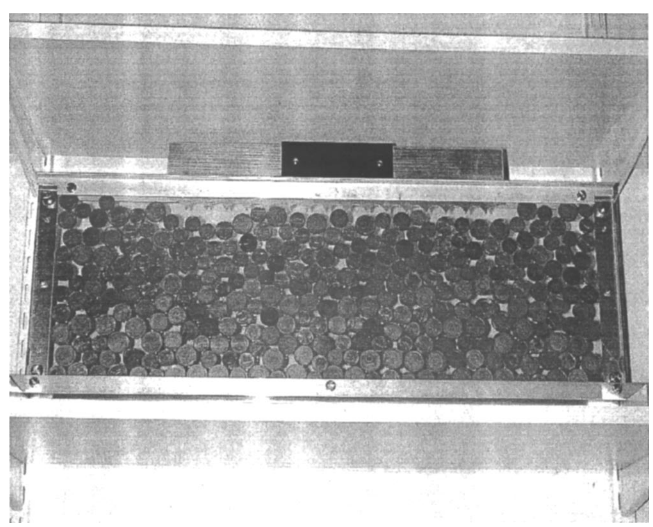

FIG. 3. Experimental apparatus.

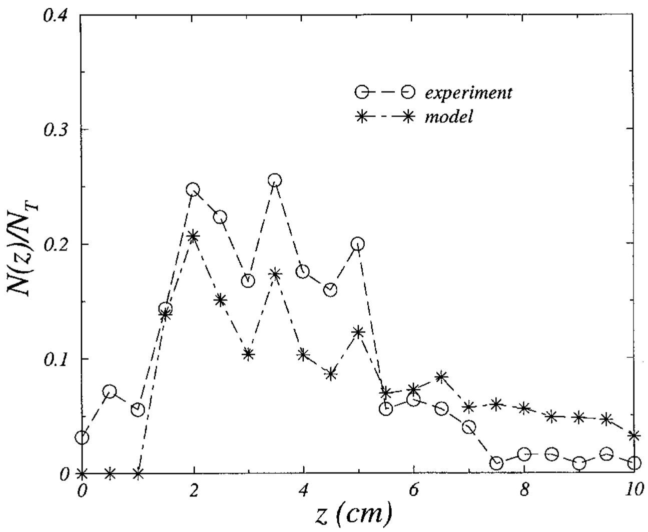

FIG. 4. Plot of the results of the experiments, and of the model, in the noncompacted medium with the force applied in the vertical direction. The model clearly reproduces the peak in the statistics but cannot reproduce the amplitude.

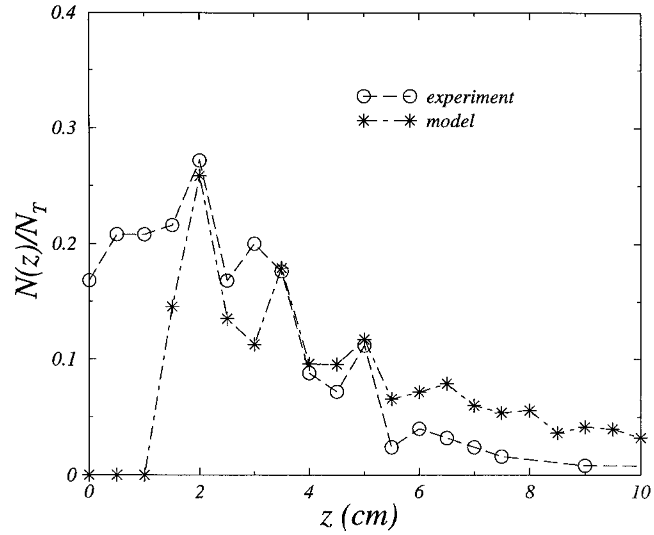

FIG. 5. Same as in Fig. 4 except that the force is applied at an angle α of 45°. In the model we use the same parameter as previously for the model, we simply take α equal to 45°. We reproduce the main characteristics of the experiments.

paction was realized tapping the box with an hammer. We stop the vibration when no significative movement occurs anymore.

The displacement is applied either normally to the surface or with an angle of 45°. When the first motion appears we measure the depth of the mobile zone as already discussed. We realized approximately 200 experiments for each configuration. After five measures made at different places of the same sample, we change the arrangement of the coins in order to make the measures on statistically independent samples. We perform these experiments on the two different types of samples presented previously, and evaluate the distribution of depth. This is simply the number of events of a given depth divided by the number of measurements. When the orientation of the applied stress is more parallel to the surface, the distribution changes and the rupture occurs close to the surface. We will discus the results of the experiments more precisely later.

The coins, which participate in the motion can be well identified. We tried to keep the displacements as small as possible. Typically they were of the order of 1 mm. The force that one needs to impose in order to see a motion fluctuates, but unfortunately we cannot measure it. The horizontal surface of the ensemble of coins cannot be defined precisely due to the finite diameter of the coins. There is also an uncertainty δ in the position of the sliding zone.

The results of the experiments made with a noncompact medium are shown in Figs. 4 and 5. Figure 4 shows the results when the stress is normal. We observe three peaks at 2, 3.5, and at 5 cm corresponding to one, two, or three average coin sizes. In the second curve the stress is orientated at an angle of 45° and the peaks are located at the same depths, but the distribution changes. More events occur close to the surface.

In both cases the number of events decreases rapidly with the depth and practically no events were measured at depths larger than 10 cm. This seems to be a consequence of the finite size of the experimental box. We will focus on this point is Sec. IV.

Another important fact is that the sliding zone is localized to a band at a depth of 2–3 particle diameters.

With the compacted medium we made the same measurements. The results are qualitatively the same as seen in Figs. 6 and 7. The number of events decreases rapidly with the depth. Here we observe essentially two peaks, and their position depends on the orientation of the displacement. When the stress is normal, the first peak is located at 3 cm and the second one at around 5 cm. If the stress is orientated by an angle of 45°, the first peak appears at 2 cm and the second at around 4 cm as seen in Fig. 6. As for the noncompacted

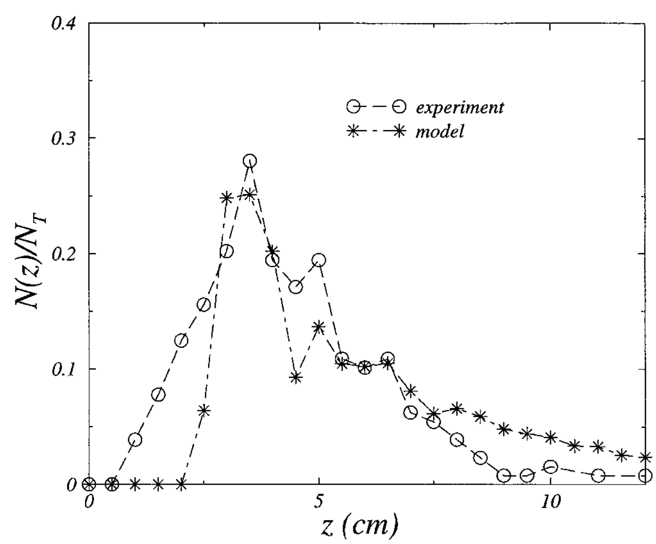

FIG. 6. Statistic of the position of the rupture. Measurements were made with a compacted medium, $\alpha=0°$. The graph shows the result of the numerical simulation.

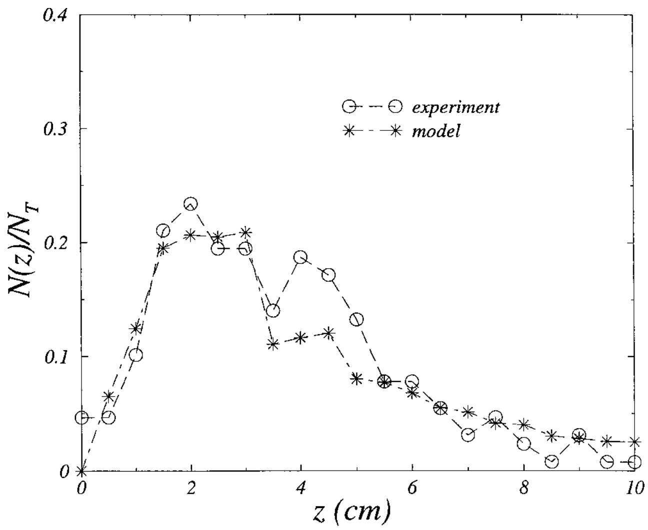

FIG. 7. The same with $\alpha = 45°$ and $\lambda = 78°$, see Table I

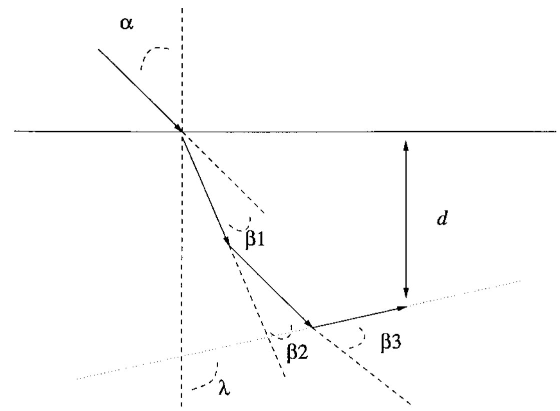

FIG. 8. Model of the stress propagation. We assume that the strain follows stochastic paths. If the path reaches the critical angle λ we consider that it can participate to the deformation and dilate the medium. The angle α can be controlled experimentally; we used both $\alpha = 0°$ and $\alpha = 45°$

medium, the movements are closer to the surface when the force is applied at 45°. The main difference with the experiments with a noncompacted medium is the mean depth of the position of the sliding zone, which is larger in the compacted case.

Summarizing, due to the discrete nature of the particles the movements are localized around particular positions, which correspond to multiples of the mean diameters of the coins. These properties will be used to test the validity of our model.

IV. MODEL AND NUMERICAL SIMULATION

In order to describe the dynamics we develop a simple numerical model, based on the idea presented in Sec. II. The model used in the simulation includes disorder.

A mechanical consequence of the granular microstructure is the fact that stresses inside the medium are obliged to pass through pointlike contacts between grains. Here we assume that the main property of the previous experimental results can be recovered focusing on the geometry of the force concentration network. The lines propagate into the medium according to simple rules given by the geometry. We use the model presented in Sec. II.

In this model we assume that the forces inside the medium deviate into the directions of the contacts. This defines an angle of deviation β, which was chosen constant as shown in Fig. 8 (45° in Sec. II). Here we choose β randomly distributed.

We will also calculate another quantity here than the one from Sec. II. Here we do not focus on the length of the path before the force turns up but on the depth where it turns back. In the simulation we need to define an angle, which indicates the position where the disks start to move. We call it the critical angle λ. In Sec. II this angle was taken to be 180°, here we will adjust its value to the experiment.

Various possibilities can be used to implement the length unit of the path. We can keep this length constant or randomly distributed. In our simulation we decided to choose a constant value equal to the mean diameter of the particles, namely, 1.6 cm.

The angle α of the orientation of the initial stress (see Fig. 8) it is fixed to 0° or to 45° depending on the experiment. This quantity is controlled externally and cannot be adjusted to fit the experiments.

Summarizing, the most important parameters are the angle on dispersion β_i that depends on the position i, the value of the critical angle λ, and the disorder amplitude l (see Fig. 9). β characterizes the direction of the stress inside the medium. It is chosen uniformly distributed around the values $+\beta_0$ and $-\beta_0$ (see Fig. 10), the width of the distribution around $+\beta_0$ and $-\beta_0$ is l, which is the only parameter in the model controlling the disorder. λ is the critical angle, when the orientation of the line reaches this value we stop the walk and measure its depth.

In order to compare the results of the simulation with the experimental data we calculate the distribution of the depth of the paths. Then we calculate the mean number of events that occur at depth $x \pm 0.5$ cm. Now we will compare the results of the model with the experimental results and discuss

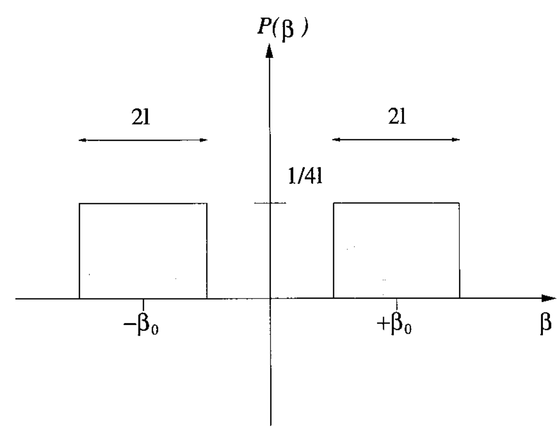

FIG. 9. Probability distribution function of β. In the model the only source of disorder is the random variable β. The strength of the disorder is controlled by l. Here $l < \beta_0$

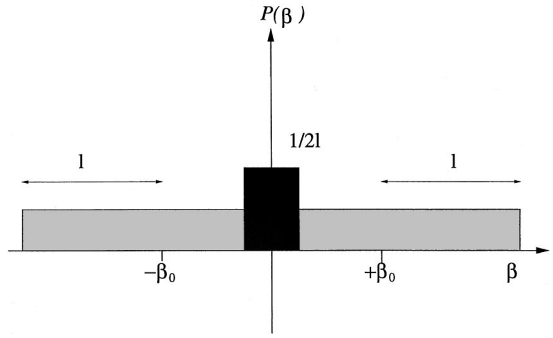

FIG. 10. Probability distribution function of β. This graph shows the distribution for $l > \beta_0$.

the value of the different parameters used to obtain the best fit.

Due to the simplicity of the model the comparison can only be qualitative. The results are shown in Figs. 4, 5, 6, and 7. We recover qualitatively the main features of the experiment. The number of events decreases rapidly with the depth, and is concentrated around two or three mean diameters of the coins. This effect is purely geometrical due to the finite size of the particle and the angle of dispersion.

We observe from Figs. 7–10 that there are systematically more shallow events in the experiment as compared to the model. In our opinion this is due to the fact that the real experimental surface is not flat and that sometimes a neighboring particle is in a higher position than the grain that is externally pushed. In that case, we observe events that the model cannot reproduce.

The value of α in the simulation is taken from the experiments i.e., 0° or 45°. Then we choose a length that corresponds to the mean diameter of the particle. In the simulation we take it 1.6 cm. The values of the other parameters are shown in Table I, β_0 is chosen such that $\beta_0 = 180°/n$, the number of contact n of a grain be between 3 and 6. The choice of l does not really influence the fit. According to the model, λ should be greater than 90° in order to give an upward force component. Our best fit gave, in some cases, smaller values. The fact that λ fits best for values less than 90° in some cases shows that the events can be generated even when the force between the centers of mass of two contacting grains has no positive upward component. We explain this fact by the observation that in our experimental case of circular particles many events are triggered by rolling, since for rolling the force that needs to be applied must not go through the center of mass, and λ can also be smaller than 90°. Rolling is not included in our model and is in fact very rare if the grains are arbitrarily shaped.

TABLE I. Parameters obtained by fitting to the experimental data.

	α	β_0	l	λ
Noncompact	0°	30°	±45°	64.3°
Noncompact	45°	30°	±45°	64.3°
Compact	0°	51°	±36°	90°
Compact	45°	51°	±36°	78°

Then we test the validity of the model. We change the parameters α (0°→45°) and we let β_0 and l unchanged. In both, the compacted and noncompacted cases, the model predicts quite well the change in the distribution, but the result for the great length disagree systematically. This systematic effect seems to be due to the finite size of the box. The longest paths, which produce the largest events, are suppressed due to the finite size of the experimental box. It would be interesting to make the same experiment with a bigger box, and to test the validity of these assumptions.

In the case where the coins are not compacted, the dispersion angle that permits to recover qualitatively the results has the value $\pi/6$ and the value of the critical angle is $\pi/2.8$. In the second experiment we work with a compacted medium, here the model gives the value $\pi/2.5$ for the deviation angle and $\pi/1.2$ for the critical angle. We summarize the results in Table I.

V. CONCLUSION

Summarizing, we have proposed a mechanism for shear bands by which grains can unlock through collective motions (dilatancy) of grains close to the center of the band. A quantitative analysis of a simplified [23] model shows this to be a plausible mechanism and justifies a width of about ten grain diameters for the mobilizable zone, i.e., the shear band. Although our argument is, of course, very simplified, we have shown that the model can reproduce qualitative experimental data. We have also considered disorder in the position of grains introducing fluctuations in the local deviation angles β. The average value of β_0, which we chose in our calculation, will also depend on the material used and will be smaller for higher polydispersity. We have disregarded the fluctuations in the forces, which are known to be very strong [21]. Since the argument is based on the existence of a force line rather than on the value of the force through it, these fluctuations might not be so important. There is, in general, also disorder in the grain sizes, and our model considers only the mean diameter of the particle. Finally, our calculation was two dimensional. A generalization of our model to three dimensions should be performed.

One could take the effect of friction into account by reweighting the paths that turn upward by a mobilization factor, for example, inversely proportional to its length. The introduction of such a factor instead of a constant static friction threshold seems justified due to the fact that the forces are in reality random. Multiplying the mobilization factor with the probability to have a path as calculated above and normalizing appropriately give a likelihood of 76% to mobilize a path within the first five layers. The above argument was made for one-half of a shear band. For the entire shear band it, therefore, seems that a width of ten grain diameters is very consistent with our picture.

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